Microeconomics

Lesson One: Choices, Preferences, and Utility

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Abstract

This note introduces the underlying framework for two models of individual decision making. The first is an *unobservable preference* based model and an *observable choice* based model. Given the emphasis mainstream economic theory places on a preference based approach, the the bulk of this note explores the former model.

To lend support to the preference model, attention is given to the conditions required on the choice model so that the two approaches are equivalent. Essentially, if an agent's choices satisfy some consistency requirements, then we are able to say that the observed choices are consistent with an economic agent with rational preferences.

The main focus of this note is on going from preferences to utility functions, and then describing the connection between the two mathematical objects. While we could 'do economics' using preferences, the consumer problems we will encounter later on will be much easier mathematically if we can use functions - especially if the functions are continuous and differentiable. As will be seen in the consumer theory notes, utility functions are the building block for all consumer problems in both microeconomic (individual and game theoretic decision problems) and macroeconomic analyses.

These notes are primarily based on lectures by Equia (MSU), Luca (U.Pittsburgh), Boarder (CalTect), and Duggan (U.Rochester) and books by Mas-Colell, Whinston, & Green; Varian; and Kreps.

Contents

1 Math Tools

1.1 Binary Relations

A binary relation is a specified relationship between two elements from two sets often denoted by an ordered pair. For example: the relation between a mother and child is a binary relation; a equation such as $y = f(x)$ is a binary relation; the fact that I live on the planet Earth is binary relation. These mathematical objects will serve as the foundation of a preferences based model of individual decision making that we are going to explore. To introduce these objects we will first define several properties and give some results about how they can be combined in order to give properties that are desirable for decision making problems.

Definition 1.1. (Binary Relation)
Suppose A and B are sets, then $R \subseteq A \times B$ is a binary relation from A to B.

Remark. We can denote the binary relation $R : A \to B$ using different notations: $R(a) = b$, $(a, b) \in R$, $R \subseteq A \times B$, and aRb. This makes obvious that functions are a type of binary relation.

Example 1.1. Weakly Greater-Than:
Let $X = \{1, 2, 3\}$, so $\geq \subset \{1, 2, 3\} \times \{1, 2, 3\}$ and $\geq = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$.

Example 1.2. Strictly Greater-Than:
Let $X = \{1, 2, 3\}$, so $> \subset \{1, 2, 3\} \times \{1, 2, 3\}$ and $> = \{\cancel{(1, 1)}, (2, 1), \cancel{(2, 2)}, (3, 1), (3, 2), \cancel{(3, 3)}\}$.

Remark. If R is a binary relation, then we refer to the 'asymetric component of R' as P, where $xRy \wedge \neg yRx \implies xPy$. In the above example, for binary relation $\geq$, the asymetric component is $>$.

Example 1.3. Strictly Less-Than:
Let $X = \{1, 2, 3\}$, so $< \subset \{1, 2, 3\} \times \{1, 2, 3\}$ and $< = \{(1, 2), (1, 3), (2, 3)\}$.

Remark. If R is a binary relation, then we refer to the 'dual of R' as R', where $xRy \implies yR'x$. In the above example, for binary relation $>$, the dual is $<$.

Definition 1.2. (Properties of Binary Relations)
A binary relation $R \subseteq X \times X$ (hereafter R on X) [1] is

1. Reflexive: $\forall x \in X, xRx$
2. Total: $\forall x, y \in X, x \neq y \implies xRy \vee yRx$
3. Complete: $\forall x, y \in X,\ xRy \vee yRx$
4. Irreflexive: $\forall x \in X, \neg\, xRx$
5. Symetric: $\forall x, y \in X, xRy \implies yRx$
6. Asymetric: $\forall x, y \in X, xRy \implies \neg\, yRx$
7. Antisymetric: $\forall x, y \in X, xRy \wedge yRx \implies x = y$
8. Transitive: $\forall x, y, z \in X,\ xRy \wedge yRz \implies xRz$
9. Negatively Transitive: $\forall x, y, z \in X, \neg\, xRy \wedge \neg\, yRz \implies \neg\, xRz$
10. Acyclic: $\forall \{x_i\}_1^n \subseteq X, x_1Rx_2 \wedge x_2Rx_3 \wedge \ldots x_{n-1}Rx_n \wedge \neg\, x_2Rx_1 \wedge \neg x_3Rx_2 \wedge \ldots \neg x_nRx_{n-1} \implies x_1Rx_n \wedge \neg x_nRx_1$

[1] Note: this is specifying that the binary relation is a self-mapping function where $X = Y$. This is by far the leading case for our purposes.

11. Negatively Acyclic: $\forall \{x_i\}_1^n \subseteq X, \neg x_1 R x_2 \wedge \neg x_2 R x_3 \wedge \ldots \neg x_{n-1} R x_n \wedge \implies \neg\, x_1 R x_n \wedge x_n R x_x 1$

Remark. Of these properties, 10 & 11 may seem odd, but each essentially states that these properties rule out the formation of a cycle of elements by the binary relation. For example, acyclicity of R on X rules out $wPxPyPzPx$, as we must have $xRz \wedge \neg\, zRp$ (i.e., xPz) which implies $\neg zPx$. Negative acyclicity is just the negative version of this.

Proposition 1. *Let R be a binary relation on X.*

1. R *is* Complete $\iff$ R *is* Total *and* Reflexive.
2. R *is* Asymetric $\iff$ R *is* Irreflexive *and* Antisymetric.
3. R *is* Irreflexive *and* Transitive $\implies$ R *is* Acyclic.
4. R *is* Reflexive *and* Negatively Transitive $\implies$ R *is* Negatively Acyclic.
5. R *is* Total *and* Transitive $\implies$ R *is* Negatively Transitive.
6. R *is* Antisymetric *and* Negatively Transitive $\implies$ R *is* Transitive.
7. R *is* Complete *and* Transitive $\implies$ R *is* Negatively Transitive.
8. R *is* Complete *and* Negatively Transitive $\implies$ R *is* Negatively Acyclic.

Proposition 2. *If P is an* Asymetric *and* Negatively Transitive *binary relation on X, then P is* Transitive, *thus P is also* Acyclic.

Proposition 3. *Let R be a binary relation on X with asymetric component P.*

1. R *is* Acyclic $\iff$ R *is* Asymetric $\wedge$ P *is* Acyclic.
2. R *is* Negatively Acyclic $\iff$ R *is* Complete $\wedge$ P *is* Acyclic.
3. R *is* Transitive $\implies$ P *is* Transitive $\implies$ P *is* Acyclic.

Remark. If R Reflexive, Total, and Transitive, then R is Negatively Acyclic. If we add Asymetric, then R is also Acyclic. The key point is that no cycles can occur for Complete and Transitive binary relations, except for equality.

1.2 Preference Relations

Here we will define prefernce relations and economic rationality to develop our model of preference based decision making. Mathematically, rational preference relations are simply a subset of the possible binary relations on a set. On this structure we apply the economic interpretations that the set of options are consumption bundles and the preference relation expresses an agent's ranking of options over that set.

Definition 1.3. (Peference Relation)
If $\succsim$ is a binary relation on X where $x \succsim y$ is interpreted as x is 'weakly prefered' to y or x is 'at least as good as' y, then $\succsim$ is a preference relation on X.

Definition 1.4. (Rational Peferncе Relation)
If $\succsim$ is a preference relation on X that is complete and transitive, then $\succsim$ is a rational preference relation on X.

Remark. Many authors combine the concept of preference relations and *rational* preference relations in their treatments; i.e., they leave the word 'rational' to be implied. The issues related to keeping this distiction in mind are covered in this note, but for the vast majority of economic applications, rationality will often be assumed implicitly.

Definition 1.5. (Asymetric and Symetric Components of $\succsim$)

1. The asymetric component of $\succsim$ is $\succ$, where $x \succ y \iff x \succsim y \wedge \neg\, y \succsim x$.
2. The symetric component of $\succsim$ is $\sim$, where $x \sim y \iff x \succsim y \wedge y \succsim x$.

Remark. We denote $\succsim$ as a weak preference relation since $x \succsim y \wedge y \succsim x$ are both possible (in this case $x \sim y$), while $\succ$ we call a strict preference relation as $x \succ y \implies \neg\, y \succ x$ (as well as $\neg x \sim y$).

Definition 1.6. (Upper and Lower Contour Sets of $\succsim$)

1. The upper contour set of $\succsim$, denoted $\succsim(x)$, is $\{y \in X \mid y \succsim x\}$.
2. The lower contour set of $\succsim$, denoted $\precsim(x)$, is $\{y \in X \mid x \succsim y\}$.

Definition 1.7. (Maximal Element)
Given $\succsim$ defined on X and $Y \subseteq X$, if $\forall y \in Y$, $x \succsim y$, then $x \in Y$ is maximal for $\succsim$ on Y.
We can denote the set of maximal elements as $M(\succsim, Y)$.

Remarks.

1. The relation $x \succsim y$ is a ranking of two options but is **not** a choice between x and y, as it is a function defined on a set.
2. Economic rationality of preferences is that the preference relation be complete and transitive.
3. Thus rationality implies no cycling of the rankings of options unless it is via the symetric component.
4. Clearly, if $\succsim$ is complete and there is no cycling, then there must be a 'highest ranked' element (this need not be unique for a weak preference), which we refer to as the maximal element.

What is missing? With rational preferences we have a weak ordering of choices that does not cycle. However, we know nothing about the intensity of the rankings; i.e., we have an ordinal ranking but not a cardnal ranking. Skipping ahead several major concepts, the most we will be able to learn about preference intensity is measuring how much an agent would pay to be indifferent between two choices.

How can we tell if an agent has rational preferences? One way is to ignore this question and simply assume that agents have rational preferences and 'start doing economics' from that point. This is essentially the approach of traditional Classical Demand Theory that is based on Utility Maximizing Agents. If we do not want to assume of rational preferences from the start, how can we 'empirically' observe whether an agent is rational? The next subsection will introduce the tools for a choice problem, and after that we will answer this question.

1.3 Choice Rules

For a well defined choice problem, we need to specify the set of options that can be choosen, the set of feasible options for a particular agent, and a choice rule that describes how the agent chooses from the feasible set.

What is a choice? In simple terms, a choice is a selection from a set of options. For example, at a restaurant you have a menu of options, and if you are going to eat, then you must make a choice about your meal (suppose that you only choose one dish per meal). If there is only one item you would ever consider ordering, then the choice is just that item. If there are four items you would consider and be just as happy with, then your choice could be any of those four items from the menu. If you would order anything off the menu and be just as happy, then your choice is the same as the menu itself. This is essentially the logic we use in contruction a representation of choice rules.

Definition 1.8. (Correspondence)
A correspondence is a mapping from a set, X, to the super set of all possible subsets of another set, Y, called the power set of Y.
We mathematically denote this as: $f\colon X \rightrightarrows Y$ or $f\colon X \to \mathcal{P}(Y)$ or $f\colon X \to 2^Y$, where $2^Y \equiv \mathcal{P}(Y)$.

Definition 1.9. (Choice Rule)
A choice rule for X is a self-mapping correspondence that is always non-empty valued.
We mathematically denote this as $\mathcal{C}\colon 2^X \setminus \emptyset \to 2^X \setminus \emptyset$, where $\forall S \subseteq X \setminus \emptyset,\ \mathcal{C}(S) \subseteq S$.

Definition 1.10. (Choice Structure)
A choice structure is a pair consisting of a set of alternatives and a choice rule.
We mathematically denote this as $(X, \mathcal{C}(\cdot))$.

Remark. The interpretation for the above is that S is a menu from X, and $\mathcal{C}(S)$ are the choices we would make from this menu. Based on earlier example, if X is all possible meals and S is all possible meals you could afford from Tres Amigos, then the set of meals you might choose are summarized by $\mathcal{C}(S)$.

Example 1.4. Let $X = \{$Tacos, Nachos, Quesadilla, Chile Rellena$\}$.
If $\mathcal{C}^i(X) = \{$Nachos, Chile Rellena$\}$, then we are saying agent i will only choose from this subset. This does **not** say that the agent chooses both options. In fact, we cannot actually say what the agent *will* choose, only that the choice will be from this subset.

Definition 1.11. (Induced Choice Rule)
A choice rule that is formed from a binary relation, R, is an induced choice rule, denoted as $\mathcal{C}_R$.
We mathematically denote this as $\mathcal{C}_R(A) = \{x \in A \mid xRy, \forall y \in A\}$.

Remark. This definition states that starting with a binary relation and a set of options, we can create the choice rule based on the binary relation.

Definition 1.12. (Rationizable)
If there exists a binary relation, $B \subseteq X \times X$, that would induce a choice rule, $\mathcal{C}(X)$, then that choice rule is rationizable by B, so $\mathcal{C}(X) = \mathcal{C}_B(X)$.

Remark. This definition is essentially the converse of the last definition: we start with a choice rule and see if there is a binary relation that would produce the rule. If we can construct the binary relation, then the choice rule can be rationalized.

What is the value of starting with choices? By starting with choices of agents rather than preferences, we are starting our theory with objects that we could actually observe. While reasonable, the idea of preferences and rationizability are strong assumptions to make about agent decision making that cannot be directly observed. A choice based approach has the extra benefit of allowing more general behavior of the agents, so in that sense the theory can start with fewer assumptions. However, for better or worse, a choice based foundation is the less traditional approach, and so is often shunted aside for a preference based approach.

2 Deducing Preferences from Choice

2.1 Introduction

What would be the best way for an economist to learn an agent's preferences? If the agent ranked every possible option and submitted this to us, then our job would be done, but this is an impossible task for the agent to do. The next best would be for us to see what an agent would choose from every possible menu, then we could figure out what preference relation would rationalize such a choice rule, but again this is an impossible task!

As economists, all we are able to do is observe actual choices that an agent makes. Our goal is to study when we can look at observed choice data and declare that the agent making the observed choices has preferences that are rational, so we can use the traditional approach to consumer theory. That is, under what assumptions on a choice rule will the binary that rationalizes the choice rule be complete and transitive?

Perhaps somewhat obvious, if we are given a choice rule that we cannot rationalize with a preference relation, then we cannot learn whether there are preferences that would induce that choice rule. That is, if we cannot backwards engineer the binary relation that would induce the choices we observe, then we cannot figure out the binary relation that characterizes the preferences of the agent making the choices [2]. Less obvious, even if we can find a preference relation that rationalized the choice rule, this preference relation may not uniquely rationalize the choice rule, so we still cannot learn the agent's preferences! As we will see, not all possible choice rules will result in preferencs that economists call rational even if there are binary relations that would rationalize the choice rule [3]. What complicates the investigation is that the observed choice data is not the actual choice set (i.e., $\mathcal{C}(X)$) but individual elements from the choice set (i.e., observe $x \in \mathcal{C}(X)$), so we can only say that these observed choices are revealed prefered but not necessarily *most prefered.*

For the remainder of this section, several definitions are given with some interpretive remarks, then three theorems are given with proofs that answer the question of when we can deduce rational preferences from observed choices.

2.2 Definitions

In this section, ee will define such concepts as the Revealed Preference Relation, 'Sen's α, β, & γ', and the Weak Axion of Revealed Preference. Remarks and examples are given.

Definition 2.1. (Revealed Preference Relation)
Given a choice structure $(X, \mathcal{C}(\cdot))$, the revealed preference relation, $\succsim^R$ is defined by:
$x \succsim^R y \iff \exists S \subseteq X \text{ s.t. } x, y \in S \land x \in \mathcal{C}(S)$.

Remarks.

1. We could alternatively say $\succsim^R$ is the revealed preference relation of $\mathcal{C}(\cdot)$, of the particular choice rule we are given.
2. Read this as if there is a menu with both x & y and we see the agent pick x, then x is (weakly) revealed prefered to y.

[2] One more time, if we are unable to construct a binary relation that would pick the same elements as the observed choices, then we are unable to talk about that binary relation because we did not figure out what it was!

[3] Perhaps a more subtle point than it would seem, there is no direct connection between a binary relation rationalizing a choice rule and that binary relation being a rational preference relation.

3. The point is that the agent chose some option, x, when the agent *could have* chosen an alternative, y, so it must be that at some level x is revealed to be at least as good as y (otherwise the agent would choose y).
4. This does not say that x will always be choosen instead of y for all menus, only that we observed one instance for the agent where it was.

Definition 2.2. (Sen's α Axiom)
If $\forall S, T \subseteq X$ s.t. $S \subseteq T, x \in \mathcal{C}(T) \cap S \implies x \in \mathcal{C}(S)$, then the choice rule $\mathcal{C}(\cdot)$ satisfies Sen's α Axiom.

Remarks.

1. Alternative: If $\forall x \in X$ s.t. $x \in S \subseteq T \wedge x \in \mathcal{C}(T), x \in \mathcal{C}(S)$, then the choice rule $\mathcal{C}(\cdot)$ satisfies Sen's α Axiom.
2. This is similar to an Independence of Irrelevant Alternatives property that will reappear in future section.
3. This property states that if an agent chooses x in from a big menu, T, then if the agent is given a smaller menu, S, that only includes items from the big menu, then the agent should again choose x.
4. For example, if given $M =${Coffee, Tea, Coke} and $\mathcal{C}(M) = \{$Coffee$\}$, then to satisfy the α Axiom, $\mathcal{C}(\{$Coffee, Coke$\}) = \{$Coffee$\}$

Definition 2.3. (Sen's γ Axiom)
If $\forall S, T \subseteq X,\ x \in \mathcal{C}(T) \cap \mathcal{C}(S) \implies x \in \mathcal{C}(S \cup T)$, then the choice rule $\mathcal{C}(\cdot)$ satisfies Sen's γ Axiom.

Remarks.

1. This axiom functions as a Extention Consistency property, so that as the menu gets bigger the agent's choices remain consistent.
2. This property states that if an agent chooses x in from every small menu, S & T, then if the agent is given a combined menu, $S \cup T$, then the agent should again choose x.
3. For example, if given $M =${Coffee, Tea, Coke} and $\mathcal{C}(M) = \{$Coffee$\}$ and $N =${Coffee, Orange Juice, Pepsi} and $\mathcal{C}(N) = \{$Coffee$\}$, then to satisfy the γ Axiom, $\mathcal{C}(\{$Coffee, Tea, Coke, Orar
$\{$Coffee$\}$.

Definition 2.4. (Sen's β Axiom)
If $\forall S, T \subseteq X,\ S \subseteq T \wedge\ \mathcal{C}(S) \cap \mathcal{C}(T) \neq \emptyset \implies \mathcal{C}(S) \subseteq \mathcal{C}(T)$, then the choice rule $\mathcal{C}(\cdot)$ satisfies Sen's β Axiom.

Remarks.

1. Alternative: If $\forall S \subseteq T \subseteq X$ s.t. $x, y \in S, x, y \in \mathcal{C}(S) \wedge\ y \in \mathcal{C}(T) \implies x \in \mathcal{C}(T)$, then the choice rule $\mathcal{C}(\cdot)$ satisfies Sen's β Axiom.
2. This is a more restrictive version of Sen's γ.
3. This property states that if an agent chooses x & y in from a small menu, S, then if y is chosen from big menu, T, then the agent should also choose x in big menu.
4. This says that if in a small menu two options are roughly equivalent and in a big menu I choose one of those options, then the other option (still roughly equivalent) should also be chosen.
5. For example, if $\mathcal{C}(\{$Coffee, Tea, Coke$\}) = \{$Coffee, Tea$\}$ and Coffee $\in \mathcal{C}(\{$Coffee, Tea, Coke, Oran
then we should expect Tea $\in \mathcal{C}(\{$Coffee, Tea, Coke, Orange Juice, Pepsi$\})$ to satisfy the β Axiom.

Definition 2.5. (Weak Axiom of Revealed Preference)
If $\forall S, T \subseteq X,\ x, y \in S \cap T \wedge\ x \in \mathcal{C}(S) \wedge\ y \in \mathcal{C}(T) \implies x \in \mathcal{C}(T)$, then the choice rule $\mathcal{C}(\cdot)$ satisfies WARP.

Remarks.

1. Alternative: If $\forall S, T \subseteq X,\ S \cap \mathcal{C}(T) \neq \emptyset \implies \mathcal{C}(S) \cap T \subseteq \mathcal{C}(T)$, then $\mathcal{C}(\cdot)$ satisfies WARP.
2. This property states that if $x\&y$ are both on two menus, $S\&T$, and x is chosen from S and y is chosen from T, then to satisfy WARP x must also be chosen from T.
3. This says that if a choice beats an alternative in one menu, but that alternative is chosen when the agent could have chosen the first choice, then to satisfy WARP it must be that the first choice could also be chosen from the second menu; i.e., it *cannot* be that the agent would *never choose* the first choice from second menu.
4. For example, if $\mathcal{C}(\{\text{Coffee, Tea, Coke}\}) = \{\text{Coffee}\}$ and Tea $\in \mathcal{C}(\{\text{Coffee, Tea, Pepsi, Orange Juice}\})$ then we should expect Coffee $\in \mathcal{C}(\{\text{Coffee, Tea, Pepsi, Orange Juice}\})$ to satisfy WARP.

2.3 Revealed Preferences Rationalize Choices

In this section we will use the preceding definitions to answer the question of when can we say that a given choice rule comes from an agent with rational preferences.

Theorem 1.
Let $\mathcal{C}(\cdot)$ be a choice rule on X.
If $\mathcal{C}(\cdot)$ is rationalized by a rational preference relation $\succsim$, then $\succsim$ is the equivalent to the revealed preference relation of $\mathcal{C}(\cdot)$; i.e., $\succsim = \succsim^{\mathcal{C}}$.

Proof.
Let $\succsim$ be a rational preference relation that rationalizes an observed choice rule, $\mathcal{C}$. Denote the induced choice rule constructed from $\succsim$ by $\mathcal{C}_{\succsim}$. Thus $\forall A \subseteq X,\ \mathcal{C}(A) = \mathcal{C}_{\succsim}(A) = \{x \in A \mid x \succsim y \forall y \in A\}$.
Suppose $x \succsim y$. WTS: $x \succsim^{\mathcal{C}} y$.
As $\succsim$ is rational, $x \succsim x$, so $x \succsim z\ \forall z \in \{x, y\},$ thus $x \in \mathcal{C}_{\succsim}(\{x, y\})$ by supposition.
By assumption $\mathcal{C} = \mathcal{C}_{\succsim}$, so $x \in \mathcal{C}(\{x, y\})$. Therefore, $x \succsim^{\mathcal{C}} y$.
Suppose $x \succsim^{\mathcal{C}} y$. WTS: $x \succsim y$.
By supposition, $\exists\ A \subseteq X$ s.t. $x, y \in A \wedge\ x \in \mathcal{C}(A)$.
By assumption $\mathcal{C} = \mathcal{C}_{\succsim}$, so $x \in \mathcal{C}_{\succsim}(A)$, so by definition $x \succsim z,\ \forall z \in A$. Recall, $y \in A$, so $x \succsim y$. □

Remark. The main intuition of the proof is that the observed choices, $\mathcal{C}(\cdot)$, are copied by the choice rule constructed from the rational preference relation that rationalizes the choices, $\mathcal{C}_{\succsim}$. The proof first shows that the constructed preferences match the revealed preferences, then shows that the revealed preferences must match the preferences constructed from the choices.

What does this tell us? The main result is that only revealed preferences can rationalize observed choices. Thus to see whether a choice rule is rationizable, we only need to check the revealed preference relation. To see whether the preferences that would induce the choices we see are rational, again we only need to check if the revealed preference relation is rational. If the revealed preference relation is not rational, then the observed choice rule does not originate from rational preferences. This almost statisfies our goal, but how do we know when a choice rule can be rationalized? The theorem above only states that '*if* we can rationalize $\mathcal{C}(\cdot)$' do we get information about the preferences. In the next part, we look for an answer to this new question.

2.4 When Can Choices Be Rationalized?

So now we know that if we can rationalize the choice rule with a revealed preference relation that is rational, then the underlying preferences of the agent must also be rational. When will this be possible?

Theorem 2.
Let X be finite with choice structure $(X, \mathcal{C}(\cdot))$.
The choice rule $\mathcal{C}(\cdot)$ is rationalizable $\iff$ the choice rule satisfies Sen's α & γ.

Proof.
Let $\mathcal{C}: 2^X \setminus \emptyset \to 2^X \setminus \emptyset$ s.t. $\mathcal{C}(S) \subseteq S, \forall S \subseteq X$. Let $\succsim^{\mathcal{C}} = \{(x,y) \in X \times X \mid x \in \mathcal{C}(\{x,y\})\}$.
($\rightarrow$) Suppose $\exists R \subseteq X \times X$ s.t. R rationalizes $\mathcal{C}(\cdot)$. Suppose $S, T \subseteq X$ s.t. $S \subseteq T$ and $x \in \mathcal{C}(T) \cap S = M(T,R) \cap S$. By supposition $\nexists y \in S$ s.t. $yRx \wedge \neg xRy$ (i.e., yPx). Thus $x \in M(S,R) = \mathcal{C}(S) \implies \alpha$. Suppose now that $x \in (\mathcal{C}(S) \cap \mathcal{C}(T)) = (M(S,R) \cap M(T,R))$. By supposition $\nexists y \in S$ s.t. $yPx \wedge \nexists z \in T$ s.t. zPx. Thus $x \in M(S \cup T, R) = \mathcal{C}(S \cup T) \implies \gamma$.
($\leftarrow$) Suppose $\mathcal{C}$ satisfies $\alpha \& \gamma$. Suppose $S \subseteq X,\ x \in \mathcal{C}(S)$ and $y \in S$. Then $\alpha \implies x \in \mathcal{C}(\{x,y\})$, so $x \succsim^{\mathcal{C}} y$. Since $x \in M(S, \succsim^{\mathcal{C}})$, $\gamma \implies x \in \mathcal{C}(\cup_j^n \{x, y_j\}) = \mathcal{C}(S)$. Thus $M(S, \succsim^{\mathcal{C}}) = \mathcal{C}(S)$, so $\mathcal{C}(\cdot)$ is rationalizable by $\succsim^{\mathcal{C}}$. □

The above theorem gives the necessary and sufficient claims to rationalize a choice function, so we have our answer to our fundamental question! If a choice rule satisfies Sen's $\alpha \& \gamma$, then we can rationalize the choice rule with the revealed preference relation, and if the revealed preference relaton is rational, then the preferences of the agent that made the choices must be rational.

However, typically we work with more restrictive claims (i.e., we work with sufficient claims that are not quite necessary [4]). For better or worse, we will rely on WARP for almost everything we care about doing.

Theorem 3.
If $\mathcal{C}(\cdot)$ is a choice rule on X, then the following are equivalent:

1. $\mathcal{C}(\cdot)$ *satisfies Sen's $\alpha \& \beta$.*
2. $\mathcal{C}(\cdot)$ *satisfies* WARP.
3. $\mathcal{C}(\cdot)$ *is rationalizable by a rational preference relation.*

Proof.
As $\gamma \implies \beta$, 1&3 are obviously equivalent by Theorem 2.
Here we only prove that $1 \implies 2$ (although this *is* biconditional).
Suppose $\mathcal{C}(\cdot)$ satisfies $\alpha \& \beta$. Suppose $x, y \in S \cap T \subseteq X,\ x \in \mathcal{C}(S),$ and $y \in \mathcal{C}(T)$. WTS $x \in \mathcal{C}(T)$.
Note $S \cap T \subseteq S$ and $x \in \mathcal{C}(S)$, so by assumption $\alpha \implies x \in \mathcal{C}(S \cap T)$. Similarly, as $S \cap T \subseteq T$ and $y \in \mathcal{C}(T),\ \alpha \implies y \in \mathcal{C}(S \cap T)$. Note $x, y \in \mathcal{C}(S \cap T)$ and $y \in \mathcal{C}(T)$, so by assumption $\beta \implies x \in \mathcal{C}(T)$. Thus we have that $x, y \in S \cap T \subseteq X \wedge x \in \mathcal{C}(S) \wedge y \in \mathcal{C}(T) \implies x \in \mathcal{C}(T)$, which is by definition WARP. □

[4] This sentence might mean the opposite of what you think: using a claim that is 'sufficient but not necessary' means we get same result but are adding restrictions we could get rid.

And just like that, we have completed our mission of describing when we can observe choices and deduce preferences and verify that the preferences are rational. These assumptions will allow us to use a preference based approach to economic applications and get the same results as if we started with observable choices.

3 The Concept of Utility

Utility is a concept that will be refererenced from now until the end of your economics career. In your mind, you may be thinking that utility can be approximated by 'happiness' or 'pleasure' from consumption. This is indeed the classic approach to the concept and may even still be seen in some introductory or intermediate books, but this is not the current understanding of the term by economists. Why not? If we think of utility as a numerical representation of happiness for an individual, then we must also answer some tough positive (empirical) questions:

- How could we measure happiness?
- What would any utility score mean for an individual?
- What would any utility score mean across multiple individuals?"

Supposing we had answers to these positive questions, then we would have to answer some tougher normative questions. One extreme example (`the Utility Monster`): Suppose we want to maximize happiness of society, if one agent always gets more utility from a good than every other distribution to other agents, should we give that agent all of the good? What if this was true for every good?

3.1 How should we think of utility?

The modern approach to the concept is that utility is a numerical representation of our preference ordering[5]. That is, utility is only used as a way to describe an agent's preference ranking of options. Thus, a utility score for a bundle is an index number of how an agent would rank that bundle relative to another bundle. For example, if my utility from a banana is five utiles and my utility from an apple is three utiles, then I prefer the banana to the apple, as the apple gives me less utility than the banana. However, this **does not** say that I get five units of happiness from the banana or that I get two extra units of happiness from a banana because the index units do not imply anything other than some ranking of my options. Further, if another agent gets seven utiles from bananas, then even this **does not** imply that this agent gets more happiness than I get from bananas. In fact, I might rank bananas higher than the other agent because two different utility scores are incomparable, *unless* both scores are from identical utility indicies which represent identical preferences.

An alternative explaination is that the difference between any two utility scores, $u(x) - u(y)$, contains no information other than either than x is more prefered or less prefered, determined by the sign of the difference (i.e., $u(x) - u(y) \gtreqless 0$). For two agents, i and j, $u^i(x) > u^j(x)$ provides no economic information at all unless $u^i(\cdot) \equiv u^j(\cdot), \forall\, x \in X$. Hopefully, the distinction between some hypothetical measuring of happiness and utility is clear.

3.2 What does it actually mean for a utility function to represent preferences?

Let's think about what we want to do. We have some rational preference relation, $\succsim \subseteq X \times X$, that creates a ranking of bundles, x, from the option space, X, through pairwise comparisons. We want to assign every option x from X a utility score, $u(x)$, with two requirements. First, we must always know the preference ranking if we know the utility

[5]Recall: preferences are binary relations over consumption bundles where the relationship specifies which item in the pair is prefered by an agent.

scores of various bundles. Second, the rule that assigns these scores is a valid function defined on a numerical representation of the option-space X.

This next point is pedantic. Mathematically, there is no problem in constructing a function from the bundle space (in terms of actual goods) to a numerical point in the real coordinate space, $\mathcal{R}^n$. However, for actually 'computing numbers', we represent the consumption space numerically as a subset of the real coordinate space by representing each bundle by a 'real vector' that corresponds the quantity of each good in the bundle. For example, the bundle (two tacos, three coffees, zero pickles) is represented as the real vector $(2, 3, 0)$.

Almost always when defining a utility function, we normally skip the intermediate step by stating that $X \subseteq \mathcal{R}^N$, where N is the number of different types of goods (i.e., the dimmension of the consumption space X), and X is all possible bundles with different quantities of goods for each good.

Definition 3.1. (Functional Representation of a Binary Relation)
A function, $f\colon X \to \mathbb{R}$, represents a binary relation, $R \subseteq X \times X$, if $\forall x, y \in X, xRy \iff f(x) \geq f(y)$.

Definition 3.2. (Utility Representation of a Preference Relation)
A utility function, $u\colon X \to \mathbb{R}$, represents a prefernce relation, $\succsim \subseteq X \times X$, if $\forall x, y \in X, x \succsim y \iff u(x) \geq u(y)$.

3.3 A Final Word on the Concept of Utility

For almost all of the economic work that lies ahead, we are going to focus on ordinal utility functions that only represent the ordering of options as dictated by the preference relation from which the function is derived. Because we ignored the *intensity* of preferences when constructing the preference relation, we again ignore intensity for utility functions. This means that any utility function that represents an agent's preferences can be transformed into a different utility function, as long as this new function preserves the original ordering.

Proposition 4. *Suppose $u\colon X \to \mathbb{R}$ is an ordinal representation of the preference relation $\succsim$ on X, if $v\colon \mathbb{R} \to \mathbb{R}$ is a monotonic function, then $v(u(X))$ is also an ordinal representation of $\succsim$ on X.*

However, suppose we had a preference relation that *did* account for intensity and we derived a utility function that represented these preferences, then we would have a *cardinal* utility function that represented the ordering and intensity of preferences. For only this type of utility function, the difference between two utility scores does contain information - specifically, *by how much* one option is prefered to another. However, it still **does not** contain information about some measure of happiness. We will only use this concept of utility function much later in the case of uncertainty.

Proposition 5. *Suppose $u\colon X \to \mathbb{R}$ is a cardinal representation of the preference relation $\succsim^C$ on X, if $w\colon \mathbb{R} \to \mathbb{R}$ is a linear function, then $w(u(X))$ also represents the same intensity as $\succsim^C$ on X.*

4 When Can Preferences Be Represented By A Utility Function?

In this section, several propositions will be given about when a preference relation can be represented by a utility function. The easiest way to summarize the results is 'If the preferences are rational & do not suddenly change *and* the option space has features of finiteness, then we can represent the preference with a function'. For most of the propositions, we will manipulate properties of the option-space, but first we show there are some necessary conditions on the preferences.

Definition 4.1. (Rational Preference Relation)
If a preference relation defined on an option-space, $\succsim \subseteq X \times X$, satisfies (1) & (2), then $\succsim$ is a rational preference relation.

1. Complete: $\forall x, y \in X,\ x \succsim y \vee y \succsim x$
2. Transitive: $\forall x, y, z \in X,\ x \succsim y \wedge y \succsim z \implies x \succsim z$

Definition 4.2. (Continuous Preference Relation)
Given a preference relation defined on an option-space, $\succsim \subseteq X \times X$, if $\forall x, y \in X$, the sets $\{y \in X \mid x \succsim y\}$ and $\{y \in X \mid y \succsim x\}$ are closed, then $\succsim$ is a continuous preference relation.

Remarks.

1. Rationality should be familiar; if not, then go back and read why it is important.
2. The second definition is a version of continuity that is equivalent (Why?[6]) to a sequence definition, so just change the relevant notation if you want a sequence definition.
3. The implication of continuity is that preferences do not suddenly 'jump' or 'switch direction' from very small changes in one of the commodities/options.

Proposition 6.
If there exists a utility function that represents a preferences relation, then the preference relation is rational.

Proof.
Suppose $\exists\, u \colon X \to \mathbb{R}$ s.t. $x \succsim y \iff u(x) \geq u(y)$. As $u(X)$ is a function on X, then $u(\cdot)$ must be defined on the entire domain, X, so $\forall x \in X, \exists\, u(x) \in \mathbb{R}$. First, by properties of real numbers, $u(x) > u(y) \vee u(x) < u(y) \vee u(x) = u(y)$. By assumption, $u(X)$ represents $\succsim$ on X, so $\succsim$ must be Complete.
Next, suppose $\exists\ x, y, z \in X$ s.t. $u(x) \geq u(y) \geq u(z)$. By properties of real numbers, $u(x) \geq u(y) \wedge u(y) \geq u(z) \implies u(x) \geq u(z)$, so by assumption, $\succsim$ must be Transitive.
Thus, $\succsim$ must be Rational. □

This proof is only one way; i.e., a rational preference relation is **not** guaranteed to have a utility representation. The standard counterexample is Lexicographical Preferences on an uncountably infinite space (e.g., the real coordinate space). The previous proposition tells us we must start with rational preferences if we are to have any hope in finding a utility representation.

The next few propositions will tell us conditions that guarantee the existance of a utility representation. These propositions will primarily concern the 'size' of the option-space. The following terms will be used in their 'mathematics' meanings, so consult a relevant reference to get the precise definitions: finite, countable, infinite, & compact.

[6]Hint: These sets contain all their limit points.

4.1 Countable Option Spaces - Easy

Proposition 7.
Suppose a rational preference relation is defined on a finite option-space, then there exists a utility representation defined on the option-space.

Proof.
Suppose $\succsim \subseteq X \times X \wedge |X| = n \in \mathbb{N}$. Consider $u\colon X \to \mathbb{R}$ s.t. $u(x) = |\{w \in X \mid x \succsim w\}|$, which by assumption $\forall x \in X, u(x) \leq n \in \mathbb{N}$. Suppose $x \succsim y$. Consider $z \in X$ s.t. $y \succsim z$, so $z \in \{w \in X \mid y \succsim w\}$. By supposition, $x \succsim z$, so $z \in \{w \in X \mid x \succsim w\}$. Then, $\{w \in X \mid y \succsim w\} \subseteq \{w \in X \mid x \succsim w\}$. Thus $u(x) = |\{w \in X \mid x \succsim w\}| \geq |\{w \in X \mid y \succsim w\}| = u(y)$.
Next, suppose $x \succ y$. This implies $x \succsim y$, so $\{w \in X \mid y \succsim w\} \subseteq \{w \in X \mid x \succsim w\}$. However, $x \in \{w \in X \mid x \succsim w\}$ but $x \notin \{w \in X \mid y \succsim w\}$, so $\{w \in X \mid y \succsim w\} \cup \{x\} \subseteq \{w \in X \mid x \succsim w\}$. Then, $|\{w \in X \mid y \succsim w\}| + \|\{x\}| \leq |\{w \in X \mid x \succsim w\}|$. Thus $u(y) + 1 \leq u(x)$, so $u(y) < u(x)$.
Therefore, by definition, $u(X)$ represents $\succsim$ on X. □

Proposition 8.
Suppose a rational preference relation is defined on a countably infinite option-space, then there exists a utility representation defined on the option-space.

Remark. The proof is simple enough but tedious (and ultimately unimportant). The intuition of the proof is that there is always another number between two real numbers (for this claim we can actually get by with just the rationals). For example, suppose we assign x to 1 and z to 0, but then we find y such that $x \succsim y \succsim z$. Then we can assign y to $\frac{1}{2}$. If we find w such that $x \succsim w \succsim y$, then assign w to $\frac{3}{4}$. And so on ...

4.2 Uncountable Option Spaces - Hard

Or, what if we do not want to make any assumptions that limit the size of our option-space? The next few results are highly mathematical. This is not to say they are overly complicated, but the required intuition is a 'very developed intuition,' as Kreps would say. Birkhoff's Theorem is the most general representation theorem that can be offered. The cost of this generality is having to find a 'countable order dense subspace' (a purely mathematical object without economic intuition).

However, if the preference relation is continuous, then we can look for much easier conditions on the option-space, using results from Debrue's Theorem. As it turns out, these 'easier conditions' are satisfied by Euclidean space (i.e., $\mathbb{R}^n$), which is exactly the space we want to use to represent the quantity of items in a bundle. The formal proofs are quite involved, so I am going to skip them completely. These results come (almost directly) from the Handbook on Rational and Social Choice.

Definition 4.3. (Order-Dense)
Given $\succsim$ on X, if $\forall x, y \in X$ s.t. $x \succ y, \exists\, z \in S \subseteq X$ s.t. $x \succsim z \succsim y$, then S is order dense in X with respect to $\succsim$.

Theorem 4. *(Birkhoff)*
If $\succsim$ is a rational preference relation defined on an option-space X that contains a countable order dense subspace with respect to $\succsim$, then there exists a utility representation of $\succsim$ on X.

Proposition 9.
If $\succsim$ on X is a continuous & rational preference relation and X is a seperable metric space, then there exists a countable order dense subspace of X with respect to $\succsim$.

Theorem 5. *(Debrue –* General*)*
If $\succsim$ is a continuous & rational preference relation and the option-space, X, is seperable and convex, then there exists a continious utility representation of $\succsim$ on X.

Theorem 6. *(Debrue –* Real Coordinate Space*)*
If $\succsim$ is a continuous & rational preference relation on $X \subseteq \mathbb{R}^N$ and X is convex, then there exists a continious utility representation of $\succsim$ on X.

Remarks.

1. The proofs of these theorems ultimately still use the fact that we can always find another number between two numbers, but given the uncountably infinite number of options, we have to be more sophisticated in finding those 'in-between' numbers.
2. For some math intuition on what's going on, we have $(X, \succsim)$, a space ordered by the preference relation, and we want to translate/embed this into another space with a linear order imposed: $(\mathbb{R}^n, \geq)$.
3. The only requirement to do the embedding is the 'countable order dense subset', so that is the assumption we make.
4. To get to the more popular Debreu Theorems, we use the assumptions that will yield a countable order dense subset, so Debrue is simply a more restrictive form of Birkhoff.

Note, we can easily have a continuous utility representation on a finite X but have $\succsim$ not be continous. Recall the Lexicographic Preferences, these preferences, by being rational, have a utility representation on finite and countably infinite option-spaces. Only on uncountably infinite option-spaces is there no utility representation.

This is all just to say: *pay attention to the assumptions and direction of the claims.*

5 Economic Properties of Preference Relations

In this section, we discuss properties of preference relations that are relevant for consumer theory. Since we are not starting consumer theory, the economic motivations will be skipped in favor of the mathematics.

To shorten the notation, the phrases 'given $(\succsim, X)$' or '$\succsim$ on X' should be construed as a preference relation defined on an option-space; i.e., $\succsim \subseteq X \times X$.

5.1 'Utility Representation' Properties

Definition 5.1. (Rational Preference Relation)
If $\succsim$ on X satisfies (1) $\&$ (2), then $\succsim$ is a rational preference relation.

1. Complete: $\forall x, y \in X,\ x \succsim y \vee y \succsim x$
2. Transitive: $\forall x, y, z \in X,\ x \succsim y \wedge y \succsim z \implies x \succsim z$

Definition 5.2. (Continuous Preference Relation)
Given $(\succsim, X)$, if $\forall x, y \in X$, the sets $\{y \in X \mid x \succsim y\}$ and $\{y \in X \mid y \succsim x\}$ are closed, then $\succsim$ is a continuous preference relation.

5.2 'More Is Better' Properties

Let $X \subseteq \mathcal{R}^N$, where $\dim(X) = N$. Let $d : X \times X \to \mathcal{R}$ be a metric defined on X.

5.2.1 Monotonicity

Definition 5.3. (Weakly Monotonic)
Given $(\succsim, X)$, if $x, y \in X$ and $x \geq y \implies x \succsim y$, then the preference relation is weakly monotonic.

Definition 5.4. (Strongly Monotonic)
Given $(\succsim, X)$, if $x, y \in X$ and $x > y \implies x \succ y$, then the preference relation is strongly monotonic[7].

Definition 5.5. (Monotonic)
Given $(\succsim, X)$, if $x, y \in X$ and $x \gg y \implies x \succ y$, then the preference relation is monotonic.

Proposition 10.
If $\succsim$ on $X \subseteq \mathcal{R}^N$ is stongly monotonic, then $\succsim$ is monotonic.

Proposition 11.
If $\succsim$ on $X \subseteq \mathcal{R}^N$ is stongly monotonic, then $\succsim$ is weakly monotonic.

Proposition 12.
If $\succsim$ on $X \subseteq \mathcal{R}^N$ is monotonic, then $\succsim$ is weakly monotonic.

5.2.2 Satiation

Definition 5.6. (Satiation)
Given $(\succsim, X)$, if $\exists x \in X$ s.t. $\forall y \in X,\ x \succsim y$, then then the preference relation is satiated at the satiation point x.

[7] Note: some sources refer to this as 'strictly monotonic'.

Definition 5.7. (Global Non-Satiation)
Given $(\succsim, X)$, if $\forall y \in X, \exists x \in X$ s.t. $x \neq y \land\ x \succsim y$, then the preference relation is globally non-statiated.

Definition 5.8. (Local Non-Satiation)
Given $(\succsim, X)$, and an ϵ-neighborhood defined by a metric $d(\cdot)$ on X, if $\forall y \in X, \exists x \in \mathsf{N}_\epsilon(y)$ s.t. $x \succ y$, then the preference relation is locally non-statiated[8].

Proposition 13.
If $\succsim$ on $X \subseteq \mathcal{R}^N$ is monotonic, then $\succsim$ is locally non-satiated.

Remark. A fun-fact from K.C. Border (CalTech) is that given a continuous, rational, and l.n.s. preference relation, $\succsim^{lns}$, then there exists a monotonic preference relation, $\succsim^m$, that will generate the same Walrasian demand correspondence. His point is that, since we will use utility and demand in consumer theory to 'do economics', there is no difference between the assumptions, so "the apparent increase in generality is illusory."

5.3 'Decreasing Marginal Return' Properties

We model the concept of decreasing marginal returns with the mathematical concept of 'convexity'. We could also characterize this as a desire for diversity or that 'a balanced bundle is best'. Convexity sits serendipitously in the intersection of properties that are economically meaningful and useful for mathematical modeling.

Let $X \subseteq \mathcal{R}^N$, where $\dim(X) = N$.

Definition 5.9. (Weak Upper Contour Set)
Given $(\succsim, X)$, the weak upper contour set of $\succsim$ is $R(x) = \{y \in X|\, y \succsim x\}$.

Definition 5.10. (Strict Upper Contour Set)
Given $(\succsim, X)$, the strict upper contour set of $\succsim$ is $P(x) = \{y \in X|\, y \succ x\}$.

Definition 5.11. (Covexity)
Given $(\succsim, X)$, if $x \succ y \implies \alpha x + (1-\alpha)y \succ y, \forall \alpha \in (0,1)$, then $\succsim$ is convex.

Definition 5.12. (Weak Covexity)
Given $(\succsim, X)$, if $x \succsim y \implies \alpha x + (1-\alpha)y \succsim y, \forall \alpha \in (0,1)$, then $\succsim$ is weakly convex.

Definition 5.13. (Strong Covexity)
Given $(\succsim, X)$, if $x \neq y \land x \succsim y \implies \alpha x + (1-\alpha)y \succ y, \forall \alpha \in (0,1)$, then $\succsim$ is strongly convex.

Proposition 14.
Suppose X is convex and rational $\succsim$ is defined on X, then $\succsim$ is weakly convex $\iff$ $R(x)$ is convex $\forall x \in X$ $\iff$ $P(x)$ is convex $\forall x \in X$.

Remark. Thus a weakly convex, rational, and continuous preference relation on a convex option-space implies that the 'better-than-sets' of a particular bundle are convex. Having this in the back of your head will make the connection to the quasi-concavity of the utility representation (Why?[9]).

Proposition 15.
Suppose $\succsim$ is defined on X, then $\succsim$ convex and upper-semicontinuous $\implies$ $\succsim$ is weakly convex.

[8] The acronym l.n.s. is used frequently.
[9] What is the definition of quasi-concave functions?

Remark. A preference relation being convex does **not** imply that it is weakly convex, despite the name. As the proposition says, this is only true if the preference relation is also 'upper-semicontinuous,' which is a weaker version of continuity. Thus, if you assume continuity, then you satisfy upper-semicontinuous.

5.4 Special Cases of Preferences

In this section, we examine two specific types of preferences that will appear in consumer theory repeatedly: Quasi-Linear Preferences and Homothetic. The previous properties are more mathematical features of an otherwise general binary relation. The special cases defined below concern *how* the goods in the bundles are prefered to one another. This will end up helping us for certain types of analysis in consumer theory.

Definition 5.14. (Quasi-Linear Preferences)
Let $x \in X$ be specified as (m, z) where m is a unidimmensional element and z is a $(N-1)$-dimmensional element from $X_{\neg 1}$; i.e., $x = (m, z) \in X$.
Given, $(\succsim, X)$, if $(1), (2),$ and (3), then $\succsim$ is quasi-linear with resprect to the first commodity.

1. $(m, z) \sim (m', z') \implies (m + \alpha, z) \sim (m' + \alpha, z'), \forall \alpha \in \mathcal{R}$
2. $(m + \delta, z) \succ (m, z), \forall \delta \in \mathcal{R}_{++}$
3. $\forall z', z'' \in X_{\neg 1}, \exists m', m'' \in X_1$ s.t. $(m', z') \sim (m'', z'')$

Remark. Quasi-linear preferences refer to preferences that feature a good that is desirable, where equal changes in the amount of the good do not affect the ranking between the other components of the bundles, and transfers in the good can be used to achieve indifference. In the definition, this good is the first component of the pair, m[10]. If we are considering quasi-linear preferences, standard notation is to have this with respect to the first good in the bundle (just for convenience). As a preview of consumer theory, typically one also normalizes the prices of all the goods so that the price of the first good is one; i.e., $p_m = 1$ (again just for convenience). This normalization also turns the 'quasi-linear' good into the numeraire good of the bundle[11].

Definition 5.15. (Homothetic Preferences)
Given $(\succsim, X)$, if $\forall x, y \in X,\ x \succsim y \implies \alpha x \succsim \alpha y,\ \forall \alpha \in \mathcal{R}_{++}$, then $\succsim$ is homothetic.

Remark. Homothetic preferences imply that the relationship between two bundles does not change from equal proportional changes in the amount of goods in the bundle. For example, if an agent prefers one can of COKE to one bottles of WATER, then the agent prefers one thousand cans of COKE to one thousand bottles of WATER. As it turns out, this will be vital for aggregating preferences of several consumers.

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[10]Think m for 'money'.
[11]The normalization and the term 'numeraire good' have nothing to do with quasi-linearity; the two concepts are just frequently used together for analytical convenience.

6 From Preference Relations to Utility Functions

In this section, we describe how the aforementioned economic properties of preference relations pass onto utility function representations. Much of this material is purely mathematical. We started with a set of options and a weak order defined on that set, $(\succsim, X)$. With Birkhoff's Theorem we know the necessary conditions to embed $(\succsim, X)$ onto $(\geq, u(X))$, a weak linear order on $u(X) \subseteq \mathcal{R}$ defined as $u\colon X \to \mathcal{R}$.

However, this is not the end of the story. Often in consumer theory, we will need the preferences to satisfy extra economic properties beyond the necessary conditions of Birkhoff. Natually, the properties on $\succsim$ have implications for the resulting utility function $u(X)$. This is what we shall explore.

6.1 Implications of 'Utility Representation' Properties

We know that the preference relation must be rational to have a utility function from a previous theorem. Since the non-cycling of rationality implies a bundle cannot be in two separate spots, we can think that rationality implies that $u(\cdot)$ is a function. Mathematical convenience is the sole reason for us to actually want a function.

If we want to take the 'standard' route and use Debrue's Theorem (RCS), then we get that there exist $u(X)$ that are continuous functions on X. Recall, the extra necessary conditions are that $\succsim$ be continous and X be a convex subset of $\mathcal{R}^N$. If we interpret the option space as the quantities of each good it is possible to consume *and* that the agent does not abruptly change tastes with small changes, then the conditions seem more palatable. I should somewhat apologize for making you read that the implications of the 'necessary conditions for utility function representation' is that we get a utility function.

6.2 Implications of 'More Is Better' Properties

The basic take-away is that *If more consumption is better, then utility increases in consumption.* For example, if an agent is monotonic in pizza, then two slices is better than one, ten is better than two, and ten-million is better than ten-thousand. Recall, utility has nothing to do with happiness or intensity of preference!

Proposition 16.
Suppose $\succsim$ on X admits a utility representation, $u(X)$, if $\succsim$ is weakly monontonic, then all $u(X)$ are non-decreasing.

Proposition 17.
Suppose $\succsim$ on X admits a utility representation, $u(X)$, if $\succsim$ is strongly monontonic, then all $u(X)$ are increasing.

Proposition 18.
Suppose $\succsim$ on X admits a utility representation, $u(X)$, if $\succsim$ is locally non-satiated, then all $u(X)$ are also locally non-satiated[12].

6.3 Implications of 'Decreasing Marginal Returns' Properties

In a cruel twist of terminology, convex prferences imply quasi-con*cave* utilities. The reason is that quasi-concave functions are defined by having *convex* upper contour sets.

[12] Replace $x \succ y$ with $u(x) > u(y)$ in the definition.

Thus convex prefernces and their utility representation both have convex upper contour sets, which is the basic idea that ties these ideas together.

Proposition 19.
Suppose $\succsim$ *on* X *admits a utility representation,* $u(X)$*, if* $\succsim$ *is convex, then all* $u(X)$ *are quasi-concave.*

Proposition 20.
Suppose $\succsim$ *on* X *admits a utility representation,* $u(X)$*, if* $\succsim$ *is strongly convex, then all* $u(X)$ *are strictly quasi-concave.*

6.4 Implications of 'Special Cases of Preferences'

The special preferences we consider are only considered because of the properties that the utility functions possess which they inherit from the preferences. The basic reason for these special preferences is analytical convenience in some optimization problem using the utility function that characterizes consumer theory. The first two special cases allow for aggregation of multiple agents, and the last is purely for ease of computation.

Definition 6.1. (Quasi-Linear Function)
If a function $u\colon X \to Y$ with $(m, z) \in X$ can be written as $u(m, z) = m + v(z)$, for some $v\colon X_{\neg 1} \to Y$, then $u(X)$ is a quasi-linear function.

Proposition 21.
Suppose $\succsim$ *on* X *admits a utility representation,* $u(X)$*, if* $\succsim$ *is a quasi-linear preference with respect to a good, then at least one* $u(X)$ *is a quasi-linear function in that good.*

Definition 6.2. (Homogeneous Function of Degree r)
Given a function $u\colon X \to \mathcal{R}$ where $X \subseteq \mathcal{R}^n$, if $\forall \alpha \in \mathcal{R}_{++},\ u(\alpha x) = \alpha^r$, then $u(X)$ is homogeneous of degress r.

Proposition 22.
Let $X \subseteq \mathcal{R}^N$*. Suppose* $\succsim$ *on* X *admits a utility representation,* $u(X)$*, if* $\succsim$ *is continuous and homothetic, then at least one* $u(X)$ *is homogeneous of degree one.*

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7 From Utility Back to Preferences

If you can believe it, sometimes economists just make assumptions on the utility function of the agent rather than the true primatives of the model. This is acceptable as long as we know that assumptions on utility functions are actually assumptions on preferences. In this section, we describe the reverse implications from the previous section. The mathematical intuition is the same: we are 'translating' from one ordered set to another, so their properties also 'translate' between them. As a rule of thumb, the assumption of a property for utility gives a stronger property on the preferences than were needed in the reverse direction.

For all of these properties, assume that $\succsim$ is rational and X fulfills either convexity or contains a countable order dense subset, so that we are guaranteed utility representation.

Proposition 23.
If there exists a continuous utility representation for $\succsim$*, then* $\succsim$ *is a continuous preference relation.*

Proposition 24.
If there exists a non-decreasing utility representation for $\succsim$*, then* $\succsim$ *is a monotonic preference relation.*

Proposition 25.
If there exists an increasing utility representation for $\succsim$*, then* $\succsim$ *is a strongly monotonic preference relation.*

Proposition 26.
If there exists a concave utility representation for $\succsim$*, then* $\succsim$ *is a convex preference relation.*

Proposition 27.
If there exists a strictly concave utility representation for $\succsim$*, then* $\succsim$ *is a stongly convex preference relation.*

Proposition 28.
If there exists a quasi-concave utility representation for $\succsim$*, then* $\succsim$ *is a convex preference relation.*

Proposition 29.
If there exists a strictly quasi-concave utility representation for $\succsim$*, then* $\succsim$ *is a strongly convex preference relation.*

Proposition 30.
If there exists a quasi-linear utility representation for $\succsim$*, then* $\succsim$ *is a quasi-linear preference relation.*

Proposition 31.
If there exists a continuous and homogeneous of degree one utility representation for $\succsim$*, then* $\succsim$ *is a continuous and homothetic preference relation.*

8 A Word On Differentiability

You may recall from the abstract that differentiability was mentioned, but nothing thusfar has been said about it. This is because it is **extremely** mathematical.

For the mildly curious, the Handbook of Utility Theory, Vol.1 includes references to Debrue and Mas-Collel for conditions that yield differentiability. This book provides a theorem: *if the preferences satisfy a 'smoothness' criterion (and other less controvertial properties), then there exists a differentiable utility function representation.* There is no comment on the reverse direction implications of assuming differentiability. This says nothing about assuming the utility function is twice continuously differentiable.

According to MWG (in chapter three), the intuition of the 'smoothness' criterion is that the "indifference sets be smooth surfaces that fit together nicely." Is this intuitive? Kreps likely does not believe so when he states he "unhappily knows of no reasonably intuitive" sufficient condition for continous differentiability, and recommends that differentiability be considered an "analytical convenience without without axiomatic basis." Rubinstein seems to want to avoid differentiability all together (to keep from going down a "mechanistic" route), but nevertheless gives a different approach (that I recommend skipping) to the question in his lecture notes.

Remark. Jumping ahead into Choice Under Uncertainty, as it turns out the vNM Expected Utility form, being a *cardinal* utility, has an easier route to differentiability assumptions. This is partly because the funtion is defined on a convex simplex of the option space, $\Delta(X)$, which appearently allows the function to be 'almost everywhere differentiable.' Essentially, convexity is just really, really useful mathematically.